# UC Irvine
## UC Irvine Previously Published Works


### Title
Random surfaces with a curvature-dependent action large-D expansion


### Permalink
https://escholarship.org/uc/item/5838g54s



### Journal
Nuclear Physics, Section B, 257(C)

### ISSN
0550-3213



### Authors
Bander, M
Itzykson, C



### Publication Date
1985

### DOI
10.1016/0550-3213(85)90362-1



### Copyright Information
This work is made available under the terms of a Creative Commons Attribution License, availalbe at
https://creativecommons.org/licenses/by/4.0/

Peer reviewed

eScholarship.org Powered by the California Digital Library
University of California

Nuclear Physics B257 [FS14] (1985) 531-542
© North-Holland Publishing Company

# RANDOM SURFACES WITH A CURVATURE-DEPENDENT ACTION LARGE-$D$ EXPANSION

Myron BANDER[1]

*Department of Physics, University of California, Irvine, California 92717, USA*

Claude ITZYKSON

*Service de Physique Théorique, CEN de Saclay, F-91191 Gif-sur-Yvette, France*

Received 23 January 1985
(Revised 11 March 1985)

The action for discretized random surfaces imbedded in a $D$-dimensional space is generalized to include curvature. The effects of this term on the Hausdorff dimension, $d_{\mathrm{H}}$, of the surface is investigated in a mean field or large-$D$ approximation. To order $1/D$ neither the area part of the action nor the curvature part change the leading result, namely $d_{\mathrm{H}} = \infty$.

## 1. Introduction

For a long time we have been aware of the connection of ordinary (nongauge) field theories to the statistics of random paths. Similar ideas have been used to make a correspondence between gauge theories and random surfaces [1-3]. In relativistic string theories that emerged from dual models surfaces also appear. A more prosaic appearance of such objects is in the study of interfaces of different thermodynamic phases and in the study of crystal growth.

For paths the action is taken to be proportional to the length of the path, the only geometrically intrinsic quantity. For surfaces the analogous quantity is the area and often the action is taken to be proportional to it. However, on a two-dimensional manifold we can define another intrinsic quantity, the curvature, $R(\boldsymbol{\alpha})$ ($\boldsymbol{\alpha}$ is a two-dimensional vector parametrizing the surface). An obvious candidate involving the curvature, namely the Einstein-Hilbert one, $\int d^2\alpha \sqrt{g}R$ ($g$ is the metric on the surface), fails as it is proportional to the Euler characteristic, $\chi$, of the surface; $\frac{1}{2}(2-\chi)$ equals the number of handles of the surface. *From now on we restrict ourselves to a study of surfaces with a fixed Euler characteristic.* Detailed results will be given for $\chi = 0$, corresponding to the topology of a torus. The generalization of the action that involves the curvature which we have in mind* is inspired by Polyakov's work.

[1] Supported in part by the National Science Foundation.

* This generalization is discussed in greater detail in another publication [4]; a summary of our results may also be found there.

In the continuum this is the Liouville action:

$$S=\mu^2\int d^2\alpha\,\sqrt{g(\alpha)}+\tfrac{1}{2}\lambda^2\int d^2\alpha\, d^2\beta\,\sqrt{g(\alpha)}\sqrt{g(\beta)}\,R(\alpha)K(\alpha,\beta)R(\beta)\,. \quad (1)$$

$\mu^2$ is an arbitrary length scale, and $K(\alpha,\beta)$ is the Green function of the covariant laplacian:

$$\frac{1}{\sqrt{g}}\partial_i g^{ij}\sqrt{g}\,\partial_j K(\alpha,\beta)=\delta^2(\alpha,\beta)\,. \quad (2)$$

In ref. [2] $\lambda^2$ is a definite constant related to the dimensionality of the imbedding space. We shall consider $\lambda^2$ as arbitrary and in fact not even restrict ourselves to a fixed sign for it. It should be noted that eq. (1) and the Liouville action agree on the classical level; quantum mechanics will depend on the weights given various configurations.

A quantity of interest in the study of random surfaces is their spatial extent. Specifically, we want to investigate the characteristic sizes of such surfaces as a function of their area. One possibility is that the size increases as some power of the surface area, $A$:

$$\langle x^2\rangle\sim A^{2/d_{\rm H}}\,, \quad (3)$$

where $\langle x^2\rangle$ is, for example, the average of the square of the distance of the surface from its center of mass. The quantity $d_{\rm H}$ is the Hausdorff or fractal dimension of the surface. For imbedding dimensions $D>2d_{\rm H}$ self-intersections are rare and as these are related to interactions, we expect such interactions to be unimportant in the large-distance or infrared limit. For random paths $d_{\rm H}=2$, and $D=4$ is an upper critical dimension for spin-field theories.

Although much work has been done in the continuum formulation, questions pertaining to the Hausdorff dimension are more easily dealt with using discretized surfaces. We may consider them as built on a fixed, regular lattice [3] or look at surfaces approximated by piecewise flat random triangles. With actions depending on the area term only, results in the first formulation yield $d_{\rm H}=4$ while Monte Carlo simulations [5] and mean field analysis [6, 7] in the latter formulation result in

$$\langle x^2\rangle\mu^2/D=b(D)\ln A\,, \quad (4)$$

with $b$ a $D$-dependent number. For large values of $D$, the mean field calculation [6] reproduces the above with

$$b(\infty)=1/4\pi\,. \quad (5)$$

Comparing these results with eq. (3) we conclude that $d_{\rm H}=\infty$*.

Will the inclusion of curvature-dependent terms modify this result? In attempt to answer this question we shall study perturbatively the effect of curvature on the

* In two dimensions such formulations result in $d_{\rm H}=4$ [14].

TABLE 1

| $D$ | $4\pi b(D)$ (numerical result, ref. [5]) | $1+2/D$ |
|---|---|---|
| 3 | $1.73 \pm 0.04$ | 1.667 |
| 4 | $1.43 \pm 0.03$ | 1.500 |
| 6 | $1.27 \pm 0.01$ | 1.333 |
| 12 | $1.16 \pm 0.03$ | 1.167 |

mean field calculation. As we shall see, consistency will demand that $\lambda^2 \sim D$ (cf. ref. [2]). The logarithmic dependence of $\langle x^2 \rangle$ on the area is left unchanged ($d_{\mathrm{H}}$ remains infinite) and we expect the curvature terms to enter the $1/D$ corrections to $b(D)$. To the same order we will also calculate the $1/D$ contributions of the area term of the action. We might hope to see a signal of a possible nonanalytic behavior of $b(D)$ (in $D$) indicating a changeover to a different dependence of size on area. For a reason, unclear to us, this hope is unfulfilled as the contribution of the curvature-dependent terms vanish to the order $1/D$. Our result is

$$b(D) = \frac{1}{4\pi}\left(1 + \frac{2}{D} + \frac{0 \cdot \lambda^2}{D^2} + \cdots\right). \tag{6}$$

The above reproduces rather well the Monte Carlo results (see table 1).

Nevertheless we expect the curvature terms to be important perhaps to the point of modifying the saddle-point itself. It was shown in ref. [6] that even in leading orders in $1/D$, fluctuations about the mean surface induce considerable curvature. We shall confirm this result. Numerical studies could clarify this point.

In sect. 2 we discuss the discretization of the action. Details of the $1/D$ calculations are presented in sect. 3. The analytic evaluation of some of the expressions encountered is carried out in the appendix.

In the process of finding a discrete form of the Polyakov-Liouville action we obtain a discrete version of the gradient and laplacian on a randomly triangulated curved surface. These agree, in the limit of a flat random lattice, with expressions obtained earlier [8, 9]. In a different context, this enables us to couple matter fields to discrete gravity, a subject with a renewed interest [10-12].

## 2. Discretization

We study surfaces made out of flat triangles joined at their edges. Those with the topology of a torus are obtained by mapping a triangulated $N \times N$ square, with opposite sides identified, into $\mathrm{R}^D$ (fig. 1). The vertices on the base square are specified by two integer coordinates $m$, $n$ with $-\frac{1}{2}(N-1) \leq m,\ n \leq \frac{1}{2}(N-1)$ (we chose $N$ to be odd). When no ambiguities are likely we will abbreviate the pair $(m, n)$

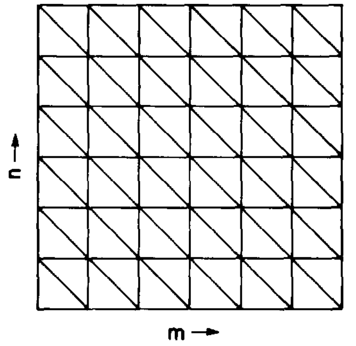


Fig. 1. The triangulated base space. Opposite sides are identified in order to obtain the topology of a torus.

by a single symbol, $i, j, k, \dots$. In general $i, j, k$ will be neighbors. The corresponding points in $\mathrm{R}^D$ will be $x_{m,n}$ or $x_i$ in abbreviated form.

The surface is piecewise flat, in that the interior of each triangle is flat. From previous works [10, 11] we know that the curvature is concentrated at the vertices of the triangles. After giving the discretized forms of the various terms entering the action, we shall also give their expansion around the situation of identical equilateral triangles; this expansion is needed for the development of the mean field approximation.

### 2.1. SURFACE AREA

For a triangle with vertices at $x_i$, $x_j$, $x_k$, with $l_{ij}^2 = (x_i - x_j)^2$, the area is given by

$$\Delta_{ijk} = \tfrac{1}{4}[2l_{ij}^2 l_{jk}^2 + 2l_{ij}^2 l_{ik}^2 + 2l_{jk}^2 l_{ik}^2 - l_{ij}^4 - l_{jk}^4 - l_{ki}^4]^{1/2} . \tag{7}$$

The total area

$$\int \sqrt{g}\, \mathrm{d}^2\alpha \to \sum_{\text{triangles}(i,j,k)} \Delta_{ijk} . \tag{8}$$

For small fluctuations around an equilateral triangle with sides of length $l$ we set

$$h_{ij} = (x_i - x_j)^2 - l^2 , \tag{9}$$

and express the area to second order in the $h$'s:

$$\Delta_{ijk} \approx \tfrac{1}{4}\sqrt{3}\, l^2 + \tfrac{1}{4}\sqrt{\tfrac{1}{3}}(h_{ij} + h_{jk} + h_{ki}) + \tfrac{1}{6}\sqrt{\tfrac{1}{3}}\frac{1}{l^2}(h_{ij}h_{jk} + h_{ij}h_{ik} + h_{jk}h_{ik} - h_{ij}^2 - h_{ik}^2 - h_{jk}^2) . \tag{10}$$

### 2.2. CURVATURE

The discrete form of the Einstein-Hilbert action has been known [10] for some time. The curvature density is related to the defect angle at the vertices of the constituent triangles (or $D$-2 simpleces for higher-dimensional manifolds). The defect angle is the difference between $2\pi$ and the sum of the angles, in our case six, of the vertices of the triangles joining at a common vertex. If $\theta_i^\alpha$, $\alpha=1,\ldots,6$ are these angles then the deficit at site $i$ is

$$\varepsilon_i = 2\pi - \sum_{\alpha=1}^{6} \theta_i^\alpha . \tag{11}$$

The discrete version of the curvature scalar follows from

$$\int d^2\alpha \sqrt{g(\alpha)}\, R(\alpha)\phi(\alpha) \to \sum_i \varepsilon_i \phi_i \tag{12}$$

for an arbitrary scalar field $\phi(\alpha)$, that takes the values $\phi_i$ at the vertices.

Up to terms linear in the $h$'s (cf. eq. (9)) we find (recall that for a torus the Euler characteristic, $\chi = (1/2\pi)\sum \varepsilon_i$, is zero)

$$\sum \varepsilon_i \phi_i \approx \frac{1}{\sqrt{3}l^2} \sum_{\text{links}(i,j)} (\phi_i + \phi_j - \phi_k - \phi_{k'}) h_{ij} . \tag{13}$$

The summation is over all links joining points $i$ and $j$; the relation of points $k$ and $k'$ to $i$ and $j$ is shown in fig. 2.

### 2.3. COVARIANT GRADIENT AND LAPLACIAN

In order to express the kernel $K$ of eq. (1) we shall need the discrete version of eq. (2). Finding the discrete form of the gradient will suffice. On each triangle we can define an induced flat metric:

$$g_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x , \qquad \alpha = 1, 2 . \tag{14}$$

For a triangle with vertices $x_i$, $x_j$ and $x_k$, the interior points maybe represented as

$$x = \alpha x_i + \beta x_j + \gamma x_k ,$$
$$\alpha + \beta + \gamma = 1 , \qquad \alpha, \beta, \gamma \geqslant 0 . \tag{15}$$

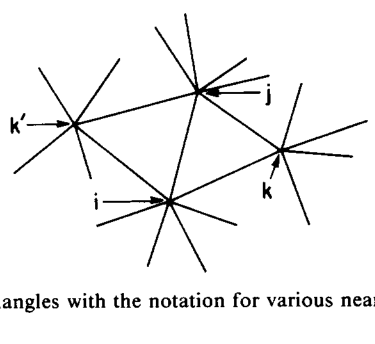


Fig. 2. Two adjacent triangles with the notation for various nearby vertices indicated.

With this parametrization the induced metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} l_{ik}^2 & l_{ik}\cdot l_{jk} \\ l_{ik}\cdot l_{jk} & l_{jk}^2 \end{pmatrix}, \tag{16}$$

with $l_{ij} = (x_i - x_j)$, etc.

A scalar field $\phi$ that takes on the values $\phi_i$ at the vertices may also be extended to the interior of the triangle; using the homogeneous coordinates of eq. (15):

$$\phi(\alpha, \beta) = \alpha\phi_i + \beta\phi_j + \gamma\phi_k . \tag{17}$$

It is straightforward to show that

$$\begin{aligned} &\tfrac{1}{2}\int \mathrm{d}^2\alpha\, \sqrt{g} g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi \\ &\qquad \to \tfrac{1}{8} \sum_{\text{triangles}(i,j,k)} \frac{1}{\Delta_{ijk}} (\phi_i l_{jk} + \phi_k l_{ij} + \phi_j l_{ki})^2 . \end{aligned} \tag{18}$$

The above reduces to a discrete version of the gradient obtained previously for the situation of random lattices on a flat plane [8, 9]. Generalizations to gradients on manifolds of higher dimension are obvious and extensions to differentials of forms higher than a scalar are given in ref. [4].

To the order of mean field calculations we shall pursue, we will need the zeroth-order (in the $h_{ij}$'s) approximation to eq. (18). This is the flat limit:

$$\begin{aligned} &\frac{1}{8}\sum_{(i,j,k)} \frac{1}{\Delta_{ijk}} (\phi_i l_{jk} + \phi_k l_{ij} + \phi_j l_{ki})^2 \\ &\quad = \tfrac{1}{2}\sqrt{\tfrac{1}{3}} \sum_{(i,j,k)} (\phi_i^2 + \phi_j^2 + \phi_k^2 - \phi_i\phi_j - \phi_i\phi_k - \phi_j\phi_k) + \cdots \\ &\quad = \tfrac{1}{2}\sqrt{\tfrac{1}{3}} \sum_{\text{links}(i,j)} (\phi_i - \phi_j)^2 + \cdots . \end{aligned} \tag{19}$$

The dots indicate higher power of $h_{ij}$.

### 2.4. PATH INTEGRALS IN DISCRETE FORM

Path integrals whose measure is determined by the action of eq. (1) may be rewritten using an auxiliary scalar field $\phi$ conjugate to the curvature:

$$\begin{aligned} \exp(-S) = &\left\{ \int [\mathrm{d}\phi] \exp\left\{ -\left[ \int \mathrm{d}^2\alpha\, \sqrt{g} + i\lambda \int \mathrm{d}^2\alpha\, \sqrt{g} R\phi \right.\right.\right. \\ &\left.\left.\left. -\tfrac{1}{2}\int \mathrm{d}^2\alpha\, \sqrt{g} g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi \right]\right\}\right\} \\ &\Big/ \left\{ \int [\mathrm{d}\phi] \exp\left[ -\tfrac{1}{2}\int \mathrm{d}^2\alpha\, \sqrt{g} g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi \right]\right\} . \end{aligned} \tag{20}$$

In previous subsections we developed an apparatus for transforming (20) to discrete form. The average of any function $F(x_i)$ of the points $x_i$ on a triangulated torus whose center of mass is held fixed is

$$\langle F\rangle=\frac{1}{Z}\int\prod \mathrm{d}x_i\,\delta(\textstyle\sum x_i)\frac{\displaystyle\int\prod \mathrm{d}\phi_i\exp(-A-i\lambda B-C)}{\displaystyle\int\prod \mathrm{d}\phi_i\exp(-C)}F(x_i)\,, \tag{21}$$

with $A$ given in eq. (8), $B$ by eq. (12) and $C$ by eq. (18). $Z$ is the normalization insuring that $\langle 1\rangle=1$. A proper definition of the measure of integration over the scalar variables $\phi_i$ requires some additional factors [4] that are triangulation dependent. These cancel in the ratio of the two $\phi$-integrations.

## 3. Mean field expansion

We shall obtain the mean field or $1/D$ results in a manner somewhat different from the one used in ref. [6] and ref. [7]. It is more in the spirit of the original application made by P. Weiss to the study of ferromagnetism and is technically more tractable than the other approaches; results are of course identical. The action is expanded around equilateral triangles of length $l$ and this length is chosen to insure that

$$\langle (x_i-x_j)^2\rangle=l^2\,, \tag{22a}$$

consequently

$$\langle h_{ij}\rangle=0\,, \tag{22b}$$

with $i, j$ nearest neighbors.

It is the area term that is used to compute $\langle (x_i-x_j)^2\rangle$. As we shall show, all other terms are of lower order in 1/D. To implement the above consistency condition we need that part of the action coming from the first two terms in eq. (10):

$$S\approx+\tfrac{1}{2}\sqrt{\tfrac{1}{3}}\mu^2\sum_{\mathrm{links}(i,j)}(x_i-x_j)^2\,. \tag{23}$$

This is the action of a free two-dimensional $D$-component field theory. The propagator is

$$\langle x^{\alpha}_{m,n},x^{\beta}_{m'n'}\rangle=\frac{\sqrt{3}\delta^{\alpha\beta}}{2\mu^2N^2}\sum_{p,q}{}'\frac{\exp(-2\pi i/N)[p(m-m')+q(n-n')]}{3-\cos(2\pi p/N)-\cos(2\pi q/N)-\cos[2\pi(p-q)/N]}\,, \tag{24}$$

$\alpha, \beta=1,\ldots,D$. The summation extends over $p, q$ integer with the mode $p=q=0$ omitted. This condition is due to the constraints on the center of mass of the surface.

From symmetry considerations we find that for $i, j$ neighbors

$$\begin{aligned}\langle (x_i - x_j)^2 \rangle &= \tfrac{1}{3}[\langle (x_{0,0} - x_{0,1})^2 \rangle + \langle (x_{0,0} - x_{1,0})^2 \rangle + \langle (x_{0,1} - x_{1,0})^2 \rangle] \\ &= D/(\sqrt{3}\mu^2)\,, \end{aligned} \tag{25}$$

and thus, according to eq. (22)

$$l^2 = D/(\sqrt{3}\mu^2)\,. \tag{26}$$

To this order in $1/D$

$$\langle x_{0,0}^2 \rangle = \frac{\sqrt{3}D}{2\mu^2 N^2} \sum{}' \frac{1}{3 - \cos(2\pi p/N) - \cos(2\pi q/N) - \cos[2\pi(p-q)/N]}\,. \tag{27}$$

For $N$ large, the summation goes over into an integration:

$$\begin{aligned}\langle\!\langle x^2 \rangle\!\rangle &= \frac{2\sqrt{3}D}{(2\pi)^2\mu^2} \int_{1/N}^{\pi} \mathrm{d}x \int_{1/N}^{\pi} \mathrm{d}y \frac{1}{3 - \cos x - \cos y - \cos(x-y)} \\ &= \frac{D}{4\pi\mu^2} \ln N^2 + \text{const.} \end{aligned} \tag{28}$$

From now on we shall use the notation $\langle\!\langle 0 \rangle\!\rangle$ as the average of some quantity over the surface, with the action expanded to whatever order necessary, while reserving the single bracket, $\langle 0 \rangle$, for an average over the free field action of eq. (23). The implications of eq. (28) were discussed in the introduction.

Before bringing in the effects of curvature we shall evaluate the $1/D$ corrections to $\langle\!\langle x^2 \rangle\!\rangle$ arising from the area part of the action. Using the last term in eq. (10) we find

$$\begin{aligned}\langle\!\langle x^2 \rangle\!\rangle = &\left\{ \langle x_{0,0}^2 \rangle - \frac{\mu^4}{6D}\left[ \sum_i \sum_{j,k} \langle x_{0,0}^2 h_{ij} h_{ik} \rangle - 2 \sum_{(i,j)} \langle x_{0,0}^2 h_{ij}^2 \rangle \right] \right\} \\ &\Big/ \left\{ 1 - \frac{\mu^4}{6D}\left[ \sum_i \sum_{j,k} \langle h_{ij} h_{ik} \rangle - 2 \sum_{(i,j)} \langle h_{ij}^2 \rangle \right] \right\}. \end{aligned} \tag{29}$$

In the summation over $j$, $k$ we include neighboring points joined by a link to $i$. The above simplifies to

$$\begin{aligned}\langle\!\langle x^2 \rangle\!\rangle = \langle x_{0,0}^2 \rangle - \frac{4\mu^4}{3D} \sum_{\alpha,\beta,\gamma=1}^{D} &\left\{ \sum_i \sum_{j,k} \langle x_{0,0}^\alpha (x_i - x_j)^\beta \rangle \langle x_{0,0}^\alpha (x_i - x_k)^\gamma \rangle \right. \\ &\times \langle (x_i - x_j)^\beta (x_i - x_k)^\gamma \rangle - 2 \sum_{(i,j)} \langle x_{0,0}^\alpha (x_i - x_j)^\beta \rangle \\ &\left. \times \langle x_{0,0}^\alpha (x_i - x_j)^\gamma \rangle \langle (x_i - x_j)^\beta (x_i - x_j)^\gamma \rangle \right\}. \end{aligned} \tag{30}$$

Details of subsequent calculations will be given in the appendix. The result is

$$\langle\!\langle x^2 \rangle\!\rangle = \frac{D}{4\pi\mu^2}\left[ 1 + \frac{2}{D} \right] \ln N^2\,. \tag{31}$$

A comparison with numerical calculations is discussed in sect. 1 and repeated in table 1.

We now turn to the effects of curvature on the extent of the surface. An integration over the auxiliary fields $\phi_i$ will result in a term quadratic in the $h_{ij}$'s with a coefficient $\sim\lambda^2/D^2$. If we wish this term to contribute to the same order in $1/D$ as the quadratic term arising from the area part of the action, we are required to take $\lambda^2$ proportional to $D$. Expanding eq. (21) to second order in $\lambda B$ we obtain

$$\ll x^2 \gg = \Big\{ \ll x^2 \gg_{\text{area}} - \frac{\lambda^2\mu^4}{2D^2} \sum_{(i_1,j_1)} \sum_{(i_2,j_2)} \langle (x_{i_1}-x_{j_1})^2 (x_{i_2}-x_{j_2})^2 x_{0,0}^2 \rangle \\ \times \langle \phi_{i_1} + \phi_{j_1} - \phi_{k_1} - \phi_{k_1'}, \phi_{i_2} + \phi_{j_2} - \phi_{k_2} - \phi_{k_2'} \rangle \Big\}, \tag{32}$$

where the relation between points $i$, $j$, $k$ and $k'$ is given in fig. 2 and the propagator of the $\phi$-field is the flat space one

$$\langle \phi_{m,n}\phi_{m',n'} \rangle = \frac{\sqrt{3}}{N^2} \sum_{p,q}{}' \frac{\exp(-2\pi i/N)[p(m-m')+q(n-n')]}{3-\cos(2\pi p/N)-\cos(2\pi q/N)-\cos[2\pi(p-q)/N]}. \tag{33}$$

The infrared divergent, but constant in $m$, $n$, term coming from the mode $p=q=0$ does not contribute to propagators of differences of the $\phi$'s that appear in eq. (3.2). We may simplify (32) to

$$\ll x^2 \gg = \ll x^2 \gg_{\text{area}} - \frac{16\lambda^2\mu^4}{D^2} \sum_{(i_1,j_1)} \sum_{(i_2,j_2)} \\ \times \langle (x_{i_1}-x_{j_1})^\alpha (x_{i_2}-x_{j_2})^\beta \rangle \langle (x_{i_1}-x_{j_1})^\alpha x_0^\gamma \rangle \langle (x_{i_2}-x_{j_2})^\beta x_0^\gamma \rangle \\ \times \langle (\phi_{i_1} + \phi_{j_1} - \phi_{k_1} - \phi_{k_1'})(\phi_{i_2} + \phi_{j_2} - \phi_{k_2} - \phi_{k_2'}) \rangle . \tag{34}$$

Again, further details may be found in the appendix. To this order the contribution of the curvature terms turns out to be zero.

We have no simple explanation for this cancellation especially in view of the fact that fluctuations around the mean field configuration induce considerable deviations from flatness, even in leading $1/D$ order. To show this we evaluate $\langle (x_{0,0}-x_{1,1})^2 \rangle$, the average distance of vertices of two triangles having a common opposite edge. In a flat case this quantity is $3l^2$. To leading order in $1/D$

$$\langle (x_{0,0}-x_{1,1})^2 \rangle / l^2 = \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi} \mathrm{d}x\, \mathrm{d}y \frac{1-\cos(x+y)}{3-\cos x-\cos y-\cos(x-y)} \\ = \frac{6\sqrt{3}}{\pi} - 2 = 1.308 . \tag{35}$$

There is a considerable deviation from flatness. This result is consistent with a similar one obtained in ref. [6].

## 4. Conclusion

To leading and next to leading orders in the mean field or $1/D$ expansion the infinite Hausdorff dimension of random surfaces is not changed by the addition to the action of curvature-dependent terms. In fact, to this order, the curvature terms do not even contribute to the logarithmic dependence of the characteristic size of the surface on its area. The noncurvature-dependent terms reproduce the Monte Carlo results.

After completion of this work we received a communication [13] pointing out problems with surfaces made out of piecewise flat random triangles. Although the triangulation we have chosen does not have the most severe problems, it is still true that for sufficiently large $N$, $\langle\langle x^N \rangle\rangle$ diverges. This result is true when the action is proportional to the area. It is unclear what effect curvature-dependent terms will have on such conclusions.

## Appendix

To insure a smoother reading of the text we have left details of the evaluation of several expressions to this appendix. Let

$$D(p,q)=3-\cos\frac{2\pi p}{N}-\cos\frac{2\pi q}{N}-\cos\frac{2\pi(p-q)}{N}. \tag{A.1}$$

### A.1. EQ. (31)

We may set $i=(m,n)$ and take $j=(m+1,n)$ and $k=(m,n+1)$. The other possibilities lead to identical results. The combinatoric factor is 6 for the first part of this expression and 3 for the second:

$$\begin{aligned}\langle\langle x^2\rangle\rangle &= \langle x_{0,0}^2\rangle-\frac{3\sqrt{3}}{\mu^2N^6}\sum\frac{e^{(-2\pi i/N)[(p_1+p_2)m+(q_1+q_2)n]}}{D(p_1,q_1)D(p_2,q_2)D(p_3,q_3)}\\ &\quad\times[(1-e^{(-2\pi i/N)p_1})(1-e^{(-2\pi i/N)q_2})(1-e^{(-2\pi i/N)p_3})(1-e^{(2\pi i/N)q_3})\\ &\quad-(1-e^{(-2\pi i/N)p_1})(1-e^{(-2\pi i/N)p_2})(1-e^{(-2\pi i/N)p_3})(1-e^{(2\pi i/N)p_3})]. \end{aligned}\tag{A.2}$$

The summation extends over the $p_i$, $q_i$ and $m$, $n$. The latter lead to a $\delta$-function restricting $(p_1, q_2)$ to be equal to $(-p_2, -q_2)$. Setting $(p_1, q_2)=(p, q)$ and $(p_3, q_3)=(P, Q)$ we obtain

$$\begin{aligned}\langle\langle x^2\rangle\rangle &= \langle x_{0,0}^2\rangle-\frac{3\sqrt{3}}{\mu^2N^4}\sum{}'\frac{1}{D^2(p,q)D(P,Q)}\\ &\quad\times\left\{(1-e^{(-2\pi i/N)}p)(1-e^{(2\pi i/N)}q)\left[1-\cos\frac{2\pi}{N}P-\cos\frac{2\pi}{N}Q+\cos\frac{2\pi}{N}(P-Q)\right]\right.\\ &\quad\left.-4\left(1-\cos\frac{2\pi}{N}p\right)\left(1-\cos\frac{2\pi}{N}P\right)\right\}. \end{aligned}\tag{A.3}$$

For large $N$ this becomes

$$\ll x^2 \gg = \langle x_{0,0}^2 \rangle - \frac{3\sqrt{3}}{\mu^2} \frac{1}{(2\pi)^4} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}X\, \mathrm{d}Y \times \frac{(1-\cos X)[(1-\mathrm{e}^{-ix})(1-\mathrm{e}^{iy}) - 4(1-\cos x)]}{[3-\cos x - \cos y - \cos(x-y)]^2 [3 - \cos X - \cos Y - \cos(X-Y)]} . \tag{A.4}$$

The range of integration of all variables is $(-\pi, \pi)$ with the region $\sqrt{x^2+y^2} < 1/N$ excluded. The terms proportional to $\ln N^2$ are easily found:

$$\ll x^2 \gg = \frac{D}{4\pi\mu^2} \ln N^2 + \frac{\sqrt{3}}{8\pi^2} \ln N^2 \int_0^{2\pi} \mathrm{d}\phi \frac{2\cos\phi - \sin\phi\cos\phi}{[1-\sin\phi\cos\phi]^2} . \tag{A.5}$$

The $\phi$-integration yields eq. (31).

### A.2. EQ. (34)

In the summations of eq. (34) we may set $i_1 = (m, n), j_1 = (m+1, n)$ and $i_2 = (m, n')$; we then have two possibilities for $j_2$, either $j_2 = (m+1, n)$ or $j_2 = (m, n+1)$ with a factor of 2. There is an overall combinatoric factor of 3. Denoting the curvature contribution to $\ll x^2 \gg$ by $\ll x^2 \gg_{\text{curv.}}$ we find

$$\begin{aligned} \ll x^2 \gg_{\text{curv.}} = \frac{-54\lambda^2}{\mu^2 D N^2} & \\ \times \sum & \frac{\exp(-2\pi i/N)[(p_1+p_3+p_4)m + (q_1+q_3+q_4)n + (p_2-p_3-p_4)m' + (q_2-q_3-q_4)n']}{D(p_1, q_1) D(p_2, q_2 (D(p_3, q_3) D(p_4, q_4)} \\ & \times \{(1-\mathrm{e}^{(-2\pi i/N)p_1})(1-\mathrm{e}^{(-2\pi i/N)p_2})|(1-\mathrm{e}^{(-2\pi i/N)p_2})|^2 \\ & \times |1+\mathrm{e}^{(-2\pi i/N)p_4} - \mathrm{e}^{(-2\pi i/N)q_4} - \mathrm{e}^{(-2\pi i/N)(p_4-q_4)}|^2 \\ & + 2(1-\mathrm{e}^{(-2\pi i/N)p_1})(1-\mathrm{e}^{(-2\pi i/N)q_2})(1-\mathrm{e}^{(-2\pi i/N)p_3})(1-\mathrm{e}^{(2\pi i/N)q_3}) \\ & \times (1+\mathrm{e}^{(-2\pi i/N)p_4} - \mathrm{e}^{(-2\pi i/N)q_4} - \mathrm{e}^{(-2\pi i/N)(p_4-q_4)}) \\ & \times (1+\mathrm{e}^{(2\pi i/N)q_4} - \mathrm{e}^{(2\pi i/N)p_4} - \mathrm{e}^{(2\pi i/N)(-p_4+q_4)})\} . \end{aligned} \tag{A.6}$$

The summation over vertices insures that $p_1+p_3+p_4 = 0 = p_2-p_3-p_4$ and likewise for the respective $q_i$'s. The log $N^2$ term will arise from the point $(p_1, q_1) = -(p_2, q_2) = 0$, which implies that $(p_3, q_3) = -(p_4, q_4)$. Converting the summations to integration the relevant part is

$$\ll x^2 \gg_{\text{curv.}} = -\frac{54\lambda^2}{\mu^2 D}\frac{1}{(2\pi)^4}\int \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}x\,\mathrm{d}y$$
$$\times\left\{\frac{3[\sin X - \sin Y - \sin(X-Y)]^2}{[3-\cos x - \cos y - \cos(x-y)]^2[3-\cos X - \cos Y - \cos(X-Y)]^2}\right.$$
$$\left.\times[1-\cos x - (1-\mathrm{e}^{-ix})(1-\mathrm{e}^{iy})]\right\}. \qquad \text{(A.7)}$$

The range of integration is the same as that in eq. (A.4). The ln $N^2$ behavior is due to the $(x, y)$ integration;

$$\ll x^2 \gg_{\text{curv.}} \sim \ln N^2 \int_0^{2\pi} \mathrm{d}\phi \frac{(\cos^2\phi - 2\sin\phi\cos\phi)}{(1-\sin\phi\cos\phi)^2}. \qquad \text{(A.8)}$$

The $\phi$-integration gives a zero result.

## References


[1] G. Parisi, Phys. Lett. 81B (1979) 357
[2] A.M. Polyakov, Phys. Lett. 103B (1981) 307
[3] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. B225 [FS9] (1983) 185; B240 [FS12] (1984) 456; Phys. Lett. B137 (1984) 83
[4] M. Bander and C. Itzykson, Proc. Seminar Series on Nonlinear equations in field theory, Paris VI and Meudon, Sept. 83–May 84 (Springer, Berlin); Saclay preprint PhT. 84-127 (1984)
[5] A. Billoire, D. Gross and E. Marinari, Phys. Lett. B139 (1984) 79
[6] D. Gross, Phys. Lett. B138 (1984) 185
[7] B. Duplantier, Phys. Lett. B141 (1984) 239
[8] N.H. Christ, R. Friedberg and T.D. Lee, Nucl. Phys. B202 (1982) 89; B210 [FS6] (1984) 310, 317
[9] C. Itzykson, Fields on a random lattice, in Progress in gauge field theory, Cargèse, 1983; Saclay preprint D Ph.G/SPT/83/148
[10] T. Regge, Nuovo Cim. 19 (1961) 551
[11] R. Friedberg and T. D. Lee, Nucl. Phys. B242 (1984) 145
[12] G. Feinberg, R. Friedberg, T. D. Lee and M. C. Ren, Nucl. Phys. B245 (1984) 343;
H.W. Hamber and R.M. Williams, Nucl. Phys. B248 (1984) 392
[13] J. Ambjørn, B. Durhuus and J. Fröhlich, B257 [FS14] (1985) 433
[14] T. Eguchi and T. Filk, Phys. Lett. 149B (1984) 381